NEW CONCEPTS IN PLASTICITY AND DEFORMATION THEORY

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This paper contains a comparative analysis of some conclusions of three new theories of plasticity, (Batdorf and Budiansky's slip theory [1]; Sanders' theory based on linear loading functions [2] and the theory proposed in [3]) and a model representation in [4].

As it is well known, the stress-strain relationships in the slip theory are quite cumbersome because of the extreme complexity of these relationships in a general case. Thus, the majority of the authors dealing with these topics limit themselves to qualitative investigations. As far as the exact analysis is concerned, outside of the conclusions of a quite general nature, some information was obtained only for the case of simple tension (compression) and subsequent application of a small tensile (compressive) additional load and twist. It was also clarified that in the case of simple tension (compression) near the point of the additional load in the "axial shear stress" plane, the plasticity curve forms an angle (plasticity angle). The sides of this angle are always tangent to the initial yield curve (von Mises ellipse). It turned out that if the additional load is directed into the exterior (in relation to the origin of the coordinate axes) angle between the tangents to the initial yield surface from the point of an additional load, then the ratio G_i of the shear stress increment to the shear strain increment is independent of the direction of the additional load inside of this angle. Besides, it coincides with the value given by an ordinary deformation theory

$$G_i = \frac{G}{1 + 3G\,[(1/E_s) - (1/E)]} \tag{0.1}$$

(E and G are the elasticity moduli; E_s is the secant modulus). If the additional load is directed into the interior angle (inside of the plasticity angle) then $G_i = G$. Finally, if the additional load is outside of these two angles, then G_i varies smoothly between the two values, even for the case of twist (orthogonal load). G_i is determined by (0.1) with $3G$

replaced by $3G/2$.

This information, however, is not sufficient to establish the objectives stated in this paper. Since the procurement of more complete information based on these relationships is connected with considerable mathematical difficulties, we make below a simplifying assumption, which will permit to obtain the required results, in particular those indicated above, by elementary means. The second of the theories, the Sanders theory, was investigated only in its general form, and as far as we know, no stress-strain relationships were obtained for this theory. These relationships are given below for plane loading paths.

As usually, the stress vector **P**, and the stress increment vector $\delta\mathbf{P}$, the strain vector Э, the plastic strain vector $Э^p$ and the plastic strain increment vector $\delta Э^p$ will denote the vectors whose components are elements of the corresponding deviation vectors. Elements with same subscripts of these deviation vectors are assumed to orientate themselves along the same unit vectors of the vector space. Since in the sequel we will consider only plane loading paths, it will be convenient to use the following notation. The direction of the additional load will be characterized by an angle α, formed by the vector $\delta\mathbf{P}$ and the unit vector **q** and perpendicular to the unit vector ρ,

$$\mathbf{q} = \delta\rho \,/\, |\,\delta\rho\,| = p\delta\rho \,/\, \delta\sigma\cos\alpha, \qquad \rho = \mathbf{P}\,/\,p$$
$$p = |\,\mathbf{P}\,| = \sqrt{\Sigma S_{ij}^2}, \qquad \delta\sigma = |\,\delta\mathbf{P}\,| = \sqrt{\Sigma\delta S_{ij}^2}, \qquad Э^p = |\,Э^p\,| = \sqrt{\Sigma Э_{ij}^{p2}} \tag{0.2}$$

The magnitude and the position of the plasticity angle (differential element of the yield curve) at point **P**, is determined by angles ϕ and ψ which form its sides with the unit vector **q**.

1. Stress-strain relationships proposed in [3] for plane loading paths. The basis of the assumptions accepted in [3] constitutes the assumptions regarding the relationships between the angle of plasticity at the point of additional loading and the vectors $\delta\mathbf{P}$ and $\delta Э^p$. It is assumed that these relationships are independent of the position of a rigidly connected system «$\delta\mathbf{P}$, $\delta Э^p$, plasticity angle » in the loading plane. Further there is the assumption regarding the locally minimal character of the variations of the yield curve along infinitesimally small segments of the loading paths. If along the $Э^p \sim p$ curve for simple loading the following condition is satisfied

$$\frac{dЭ^p}{dp} = A\left(\varphi + \tfrac{1}{2}\sin 2\varphi\right) \tag{1.1}$$

then the following is obtained:

$$\delta Э^p = A\left\{\rho\delta p\left[(\alpha+\varphi)+\frac{\sin(\alpha+\varphi)}{\sin\alpha}\sin\varphi\right]+\right.$$
$$\left.+p\delta\rho\left[(\alpha+\varphi)-\frac{\sin(\alpha+\varphi)}{\cos\alpha}\cos\varphi\right]\right\} \quad \text{for} \quad -\varphi\leqslant\alpha\leqslant\psi \qquad (1.2)$$

$$\delta Э^p = A\left\{\rho\delta p\left[(\varphi+\psi)+\frac{\sin(\varphi+\psi)}{\sin\alpha}\sin(\alpha+\varphi-\psi)\right]+\right.$$
$$\left.+p\delta\rho\left[(\varphi+\psi)-\frac{\sin(\varphi+\psi)}{\cos\alpha}\cos(\alpha+\varphi-\psi)\right]\right\} \quad \text{for} \quad \alpha\geqslant\psi$$

The quantities ϕ and ψ in these expressions are arbitrary; thus, a possibility exists for an investigation of the relationships between the theory and experiments for various modes of variations of the plasticity angle during the loading process.

2. Derivation of stress-strain relationships for the Sanders theory for the case of plane loading paths. The basic assumptions in the Sanders theory for the plane loading can be formulated as follows:

(a) For an arbitrary state **P**, arrived at by a given loading process, there exists a closed curve (yield curve) which is an envelope of a plane family of straight lines (lines of plasticity). This curve is such that an arbitrary path from **P** in the interior of this curve or along this curve (and only such path) results only nn an elastic deformation of the material.

(b) In the process of plastic deformations plasticity lines can move only away from the origin of the coordinate axes in a translatory motion (parallel to themselves). Besides, only those lines are moving, which have a common point with the stress vector.

(c) During translatory motion of a given plasticity line by a magnitude dh the plastic strain increases by an amount

$$dЭ_0^p = \Phi(h)\,dh\,\mathbf{n} \qquad (2.1)$$

where h is the distance from the origin to the given line, $\mathbf{n}$ is a unit vectorof the normal to this line in the loading plane.

(d) The total plastic strain $\delta Э^p$, produced by the additional load $\delta\mathbf{P}$ is the sum of the plastic strains $dЭ_0{}^p$ contributed by the displacements of the individual plasticity lines.

(e) Volume movement of is elastic.

Let now an additional loading $\delta\mathbf{P}$ act from some state **P**. This incremental load causes to move such plasticity lines bb, which form an angle λ with a unit vector $\mathbf{q}$, within the limits (Fig. 1):

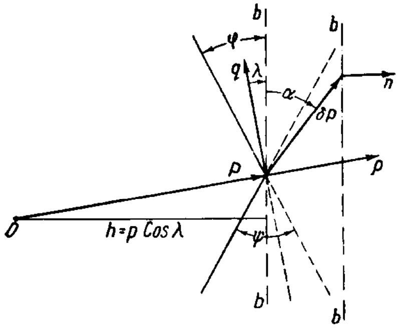

Fig. 1.

$$-\varphi \leqslant \lambda \leqslant \alpha \quad \text{for} \quad -\varphi \leqslant \alpha \leqslant \psi$$
$$-\varphi \leqslant \lambda \leqslant \psi \quad \text{for} \quad \alpha \geqslant \psi \tag{2.2}$$

From Fig. 1 we find

$$h = p \cos \lambda$$
$$dh = \delta\sigma \sin(\alpha - \lambda), \quad \mathbf{n} = \mathbf{p} \cos \lambda - \mathbf{q} \sin \lambda \tag{2.3}$$

Because of the assumptions made above, the total plastic strain increment is

$$\delta\mathbf{Э}^p = \delta\sigma \int_{-\varphi}^{\varkappa} \Phi(h) \sin(\alpha - \lambda) [\mathbf{p} \cos \lambda - \mathbf{q} \sin \lambda] \, d\lambda \tag{2.4}$$

where

$$\varkappa = \begin{cases} \alpha & \text{for } -\varphi \leqslant \alpha \leqslant \psi \\ \psi & \text{for } \quad \alpha \geqslant \psi \end{cases} \tag{2.5}$$

The assumption (b) supplies a construction method for the plasticity angles for any point of an arbitrary loading path (method of external tangents), (Fig. 2). This method also permits the determination of the angles ϕ and ψ which appear in (2.4) and (2.5). The unknown function Φ can be determined experimentally from a simple loading. If we put

$$\Phi(h) = \Phi(p \cos \lambda) = A = \text{const} \tag{2.6}$$

then from (2.4) and (2.5) for simple loading

$$\varphi = \psi, \quad \alpha = {}^1/{}_2\pi, \qquad \delta\sigma = dp$$
$$|\delta\mathbf{Э}^p| = \delta|\mathbf{Э}^p| = \delta Э^p$$

on the $Э^p \sim p$ curve the same conditions as accepted in the previous section [formula (1.1)] must be satisfied.

For a general case of loading we have

$$\delta\partial^p = \frac{A\delta\sigma}{2}\{[\sin\alpha(\lambda + \tfrac{1}{2}\sin 2\lambda) + \tfrac{1}{2}\cos\alpha\cos 2\lambda]\,\mathbf{p} + [\tfrac{1}{2}\sin\alpha\cos 2\lambda + \cos\alpha(\lambda - \tfrac{1}{2}\sin 2\lambda)]\,\mathbf{q}\}_{-\varphi}^{\varkappa} \qquad (2.7)$$

If we now utilize the first formula in (0.2) and substitute the limits of integration, taking into account (2.5), we arrive at the same expressions as shown in (1.2). In this case, however, ϕ and ψ are completely determined by the method of external tangents.

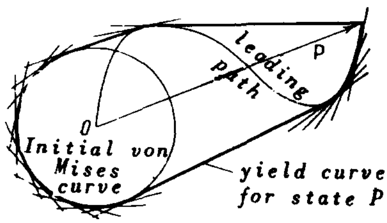

Fig. 2.

3. Derivation of the stress-strain relationships for the slip theory based on a plane body model. The assumptions of this theory are usually formulated in terms of plasticity of microstructure of the material. However, these assumptions permit of interpretations applicable for continuous media in the following way.

(a) Plastic deformation in the neighborhood of a given point in a material is a consequence of irreversible slips along some planes passing through this point.

(b) The irreversible slips occur only in those planes which contain at least one direction along which the component of the tangential stress τ, of a given plane exceeds some constant value, τ_s, and also exceeds all its previous values.

(c) Along every such direction in a given plane plastic slip γ_0^P occurs, whose magnitude depends on τ only.

(d) The total plastic strain in the neighborhood of some point of a material is the sum of all strains of the irreversible slips along all directions in all planes.

(e) Volume deformation is elastic.

Below we will investigate the consequences of this system when applied to a "plane body" model. "Plane body" model means a material which deforms in the plane of the application of the load only. The slips mentioned in (a) can, therefore, occur only along the planes perpendicular to the planes of application of the loads. For a given material the stress-strain state is determined by the stress and strain components (σ_x, σ_y, τ_{xy}, ϵ_x, ϵ_y, γ_{xy}) with respect to some fixed axes x and y in the plane of application of the load. The vector stress and strain components of deviation, in this case, are determined by

$$S_{xx} = \sigma_x - \tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(\sigma_x - \sigma_y), \quad Э_{xx} = \varepsilon_x - \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) = \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)$$
$$S_{yy} = \sigma_y - \tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(\sigma_y - \sigma_x), \quad Э_{yy} = \varepsilon_y - \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) = \tfrac{1}{2}(\varepsilon_y - \varepsilon_x)$$
$$S_{xy} = S_{yx} = \tau_{xy}, \qquad Э_{xy} = Э_{yx} = \tfrac{1}{2}\gamma_{xy} \tag{3.1}$$

If for the three-dimensional material the stress-strain vector space is in general nine-dimensional, then for the material considered here it is four-dimensional. Moreover, since the following must be satisfied for any loading

$$S_{xx} + S_{yy} = 0, \qquad S_{xy} = S_{yx}, \qquad Э_{xx} + Э_{yy} = 0, \qquad Э_{xy} = Э_{yx} \tag{3.2}$$

the loading and deformation paths lie in the same two-dimensional plane. The vectors $\mathbf{P}$ and $Э$, therefore, must be determined by the two fixed unit vectors of this two dimensional plane, for the vectors $\mathbf{P}$ and $Э$, are determined by the same rules as for a three-dimensional body. Let $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$ be unit normal vectors of this space. From the definition and (3.2) we have

$$\mathbf{P} = S_{xx}\mathbf{k}_1 + S_{yy}\mathbf{k}_2 + S_{xy}\mathbf{k}_3 + S_{yx}\mathbf{k}_4 = S_{xx}(\mathbf{k}_1 - \mathbf{k}_2) + S_{xy}(\mathbf{k}_3 + \mathbf{k}_4)$$
$$Э = Э_{xx}\mathbf{k}_1 + Э_{yy}\mathbf{k}_2 + Э_{xy}\mathbf{k}_3 + Э_{yx}\mathbf{k}_4 = Э_{xx}(\mathbf{k}_1 - \mathbf{k}_2) + Э_{xy}(\mathbf{k}_3 + \mathbf{k}_4) \tag{3.3}$$

let $\mathbf{k}_1 - \mathbf{k}_2 = \sqrt{2}\ \mathbf{i}$, $\mathbf{k}_3 + \mathbf{k}_4 = \sqrt{2}\ \mathbf{j}$. It is easy to see that the vectors $\mathbf{i}$ and $\mathbf{j}$ are orthogonal unit vectors, and we have

$$\mathbf{P} = \sqrt{2}\,[S_{xx}\mathbf{i} + S_{xy}\mathbf{j}], \qquad Э = \sqrt{2}\,[Э_{xx}\mathbf{i} + Э_{xy}\mathbf{j}] \tag{3.4}$$

Precisely in the same manner the components of the deviator vectors $\delta\mathbf{P}$, $Э^p$ and $\delta Э^p$ are expressed by $\mathbf{i}$ and $\mathbf{j}$.

Let a sample made out of this material be in some state of stress. In this case the following shear stress occurs at some angle ω to the x-direction

$$\tau = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\omega + \tau_{xy}\cos 2\omega = S_{xx}\sin 2\omega + S_{xy}\cos 2\omega \tag{3.5}$$

Along the directions where τ exceeds a certain constant value τ_s and all its preceding values in accordance with (b), we will have plastic slip

$$\gamma_0{}^p = F(\tau) \tag{3.6}$$

which produces unit plastic deformation along x and y-axes. This plastic deformation is easily determined from the geometric considerations taking into account the assumption (e), and it is

$$\varepsilon_x^{p0} = \frac{1}{2}\gamma_0^p \sin 2\omega = \frac{1}{2}F(\tau)\sin 2\omega \qquad \varepsilon_y^{p0} = -\varepsilon_x^{p0} \qquad (3.7)$$
$$\gamma_{xy}^{p0} = \gamma_0^p \cos 2\omega = F(\tau)\cos 2\omega$$

The total plastic strain along the x, y-axes is found by summing over all angles ω where τ satisfies all the above mentioned conditions. All admissible angles ω are included between $\pi/2$ and $-\pi/2$. Since, however all plastic strains which occur as a result of the two perpendicular slips are equal, we can limit ourselves to the angles $0 < \omega < \pi/2$, and then double the resultant values. Taking this circumstance into account and the second formula in (3.4) we have

$$Э^p = \sqrt{2}\int F(\tau)\,[\sin 2\omega\mathbf{i} + \cos 2\omega\mathbf{j}]\,d\omega \qquad (3.8)$$

If an incremental load is added to some state of stress, then, using the analogous arguments, it can be shown that the corresponding plastic strain increments are determined by

$$\delta Э^p = \sqrt{2}\int F'(\tau)\,d\tau\ \ [\sin 2\omega\mathbf{i} + \cos 2\omega\mathbf{j}]\,d\omega \qquad (3.9)$$

where

$$F'(\tau) = \frac{dF(\tau)}{d\tau}, \qquad \delta\tau = \delta S_{xx}\sin 2\omega + \delta S_{xy}\cos 2\omega \qquad (3.10)$$

Let us denote by β an angle formed by the vectors $\mathbf{P}$ and $\mathbf{i}$, (Fig.3), in the loading plane. Then

$$\mathbf{i} = \boldsymbol{\rho}\cos\beta - \mathbf{q}\sin\beta, \qquad \mathbf{j} = \boldsymbol{\rho}\sin\beta + q\cos\beta \qquad (3.11)$$
$$S_{xx} = \frac{P}{\sqrt{2}}\cos\beta, \qquad S_{xy} = \frac{P}{\sqrt{2}}\sin\beta$$
$$\delta S_{xx} = \frac{\delta\sigma}{\sqrt{2}}\sin(\alpha-\beta), \qquad \delta S_{xy} = \frac{\delta\sigma}{\sqrt{2}}\cos(\alpha-\beta)$$

In this new notation τ and $\delta\tau$ are expressed as follows:

$$\tau = \frac{1}{\sqrt{2}}p\sin(\beta+2\omega), \qquad \delta\tau = \frac{1}{\sqrt{2}}\delta\sigma\cos(\alpha-\beta-2\omega) \qquad (3.12)$$

For simple loading β is constant. This is also true for all incremental loadings from some arbitrary state of stress. Thus, introducing the notation

$$\beta + 2\omega = \frac{1}{2}\pi + \lambda \qquad (3.13)$$

and passing in (3.8) and (3.9) from unit vectors $\mathbf{i}$ and $\mathbf{j}$ to ρ and $\mathbf{q}$, we obtain

$$Э^p = \frac{\sqrt{2}}{2} \int F(\tau) [\boldsymbol{\rho} \cos\lambda - \mathbf{q} \sin\lambda] \, d\lambda \qquad (3.14)$$

$$\delta Э^p = \frac{\delta\sigma}{2} \int F'(\tau) \sin(\alpha - \lambda) [\boldsymbol{\rho} \cos\lambda - \mathbf{q} \sin\lambda] \, d\lambda \qquad (3.15)$$

Notice that (3.14) is valid for simple loadings and (3.15) - always. In these formulas $\tau = (p/\sqrt{2}) \cos\lambda$. Denoting as before [first formula in (2.3)], $h = p \cos\lambda$, and setting further

$$F(\tau) = F\left(\frac{h}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} \chi(h) \qquad (3.16)$$

we obtain

$$F'(\tau) = \frac{dF(\tau)}{d\tau} =$$

$$= \frac{2}{\sqrt{2}} \frac{d\chi(h)}{d\tau} = 2 \frac{d\chi(h)}{dh} = 2\,\Phi(h) \qquad (3.17)$$

The limits of integration in (3.14) and (3.15) are not determined. Let us find these limits for simple loading followed by an additional incremental load.

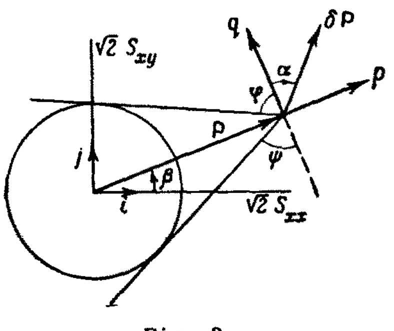

Fig. 3.

Let some simple loading take place from the undeformed state. The first plastic deformations will appear when the maximum value of $\tau = (p/\sqrt{2}) \cos\lambda$ will reach some limiting value $\tau_s = p_s/\sqrt{2}$, i.e. when

$$p = |\mathbf{P}| = p_s \qquad (3.18)$$

Thus, the initial yield curve in the loading plane belongs to a family of circles with radius p_s. With increasing p_s plastic slips are spreading out in a pencil of lines. The limiting rays are determined by the condition

$$\cos\lambda = \frac{p_s}{p} \qquad (3.19)$$

Denoting by ϕ the angle formed by $\mathbf{q}$ with the nearest tangent from $\mathbf{P}$ to the initial yield circle (Fig. 3), we obtain $p_s/p = \cos\phi$, and consequently, (3.19) means that

$$\lambda = \pm \varphi \tag{3.20}$$

Thus the limits of integration in (3.14) are determined by

$$-\varphi \leqslant \lambda \leqslant \varphi \tag{3.21}$$

For simple loading the plastic strain vector must be directed along the constant vector ρ. To make (3.14) satisfy this condition it is necessary to assume that functions $F(r)$ and $\chi(h)$ are even functions. It is easy to see that under this assumption (3.14), together with (3.16), determine the $\partial^p \sim p$ curve for simple loadings, i.e.

$$\partial^p = 2\int_0^{\varphi} \chi(h) \cos\lambda \, d\lambda \tag{3.22}$$

If at the end of a simple loading some arbitrary small load is added, then the additional plastic slip can occur only along those directions where they had place at any previous instant, i.e. in the pencil (3.22), and only along those directions of the pencil, where $\delta r \geqslant 0$, i.e. where $\alpha \geqslant \lambda$. Consequently, if α is smaller than a minimum possible value of $\lambda = -\phi$, then the so directed additional load is associated with elastic deformation only. This means that at the end of a simple loading the yield curve forms an angle, the sides of which are touching the initial plasticity circle (von Mises circle). Next, if α is limited by $-\phi < \alpha \leqslant \phi$, then the pencil, in which the additional slips are taking place, and consequently the limits of integration in (3.22) are given by

$$-\varphi \leqslant \lambda \leqslant \alpha \tag{3.23}$$

If $\alpha > \phi$, then plastic deformation will be in the whole pencil (3.21).

Thus, for the relationships between an incremental loading from the end of some simple loading and the vector $\delta\partial^p$ we have the same expressions as in the Sanders theory, [formulas (2.4) and (2.5), where $\phi = \psi$]. The only difference is that the vectors appearing in these formulas determine the stress-strain relationships for two-dimensional materials, If, as previously, we will characterize the magnitude and position of the plasticity angle in an arbitrary state by the angles ϕ and ψ, then, in the general case, the additional loading from an arbitrary state will have the form given by (2.4) and (2.5), and conditions (1.1) will be replaced by (1.2). The values of ϕ and ψ in these formulas have to be considered as determined. Besides, the explicit formulas found for the variation of ϕ and ψ for simple loadings by the external tangents method suggest that a similar method may be applicable for a wider class of loading paths.

We shall now make an assumption (mentioned in the introduction) that the relationships obtained above are also valid for a three-dimensional body under plane loading. This means that we will consider the vectors entering into these relationships as being in a nine-dimensional vector space of real materials. Since exactly the same relationships were obtained in this case as those found at the beginning of this paper (this fact will be demonstrated below), we have some reason to expect that the stress-strain relationships for the class of the loadings considered here, are either a direct consequence of the initial system of the assumptions, or are sufficiently close to it. Thus, most probably, at least for an additional loading at the end of simple loading, the conclusions of the slip theory will agree with the Sanders theory.

We will now show how from (2.4) and (2.5) for $\phi = \psi$, $\Phi = \chi'$ and from (3.22) the results mentioned in the introduction follow. If the additional loading at the end of simple loading has the components $\delta\sigma_q = \delta\sigma \cos a$ along $\mathbf{q}$, then the associated plastic strain increments have also a component $\delta\vartheta_q^p$ in the same direction, whereby

$$\frac{\delta\vartheta_q^p}{\delta\sigma_q} = -\frac{1}{\cos\alpha}\int_{-\varphi}^{\varkappa}\chi'(h)\sin(\alpha-\lambda)\sin\lambda\, d\lambda \tag{3.24}$$

Since

$$\chi'(h) = \frac{d\chi(h)}{dh} = -\frac{1}{p\sin\lambda}\frac{d\chi(h)}{d\lambda} \tag{3.25}$$

then integrating (3.24) by parts, we obtain

$$\frac{\delta\vartheta_q^p}{\delta\sigma_q} = \frac{\operatorname{tg}\alpha}{p}\Big[\chi\cos\lambda + \int\chi\sin\lambda\, d\lambda\Big]_{-\varphi}^{\varkappa} + \frac{1}{p}\Big[\chi\sin\lambda + \int\chi\cos\lambda\, d\lambda\Big]_{-\varphi}^{\varkappa} \tag{3.26}$$

If $\kappa = \phi$ (the additonal loading being directed into the interior of the angle $a \geqslant \phi$), then the first bracket and the first term of the second bracket are equal to zero. This is so because $\chi(h) = \chi\ (p \cos \lambda)$ is an even function and $\chi\ (p \cos \phi) = \chi\ (p_s) = 0$.

If $\kappa = 0$ (an orthogonal additional load $a = 0$), then the terms in the first bracket reduce to zero, as well as the first term of the second bracket, since fo $\lambda = \phi$, $\chi = 0$, and for $\lambda = 0$, $\sin\lambda = 0$.

Because of all this and (3.22) we have

$$\frac{\delta\vartheta_q^p}{\delta\sigma_q} = \frac{\vartheta^p}{p} \quad \text{for } \varkappa = \varphi, \qquad \frac{\delta\vartheta_q^p}{\delta\sigma_q} = \frac{1}{2}\frac{\vartheta^p}{p} \quad \text{for } \varkappa = 0 \tag{3.27}$$

Suppose that, before an additional load is applied, a body is subjected to simple tension (compression), then

$$\vartheta^p = \sqrt{\Sigma\vartheta_{ij}^{p2}} = \sqrt{\frac{3}{2}}\,\vartheta_{xx}^p = \sqrt{\frac{3}{2}}\,\varepsilon_x^p,\ p = \sqrt{\Sigma S_{ij}^2} = \sqrt{\frac{2}{3}}\,\sigma_x$$

Consequently

$$\frac{\partial^p}{p} = \frac{3}{2} = \frac{\varepsilon_x^p}{\sigma_x} = \frac{3}{2}\left(\frac{1}{E_s} - \frac{1}{E}\right) \tag{3.28}$$

If the additional load is tension (compression) and twist, then

$$\delta \partial_q^p / \delta \sigma_q = (1/2)(\delta \gamma_{xy}^p / \delta \tau_{xy})$$

Since $G_i = \delta \tau_{xy} / \delta \gamma_{xy}$, then

$$G_i = \frac{G}{1 + G(\delta \gamma_{xy}^p / \delta \tau_{xy})} \tag{3.29}$$

These results completely coincide with the results described in the introduction. It is also easy to see that the initial yield curve in the σ_x τ_{xy} plane will be the von Mises ellipse. The sides of the plasticity angle will touch this yield curve for the point of the additional loading σ_x.

4. Rabotnov's model which illustrates strain hardening properties of materials. The phenomena which take place in strain hardening materials may be possibly investigated qualitatively using some models, in particular, using an example of pure bending of thin-walled cylinder of elasto-ideally plastic material.

Denoting

$$\mathbf{M} = \frac{1}{R^2 t \sigma_s}[M_x \mathbf{i} + M_y \mathbf{j}], \qquad \mathbf{N} = \frac{ER}{\sigma_s}[\varkappa_x \mathbf{i} + \varkappa_y \mathbf{j}] \tag{4.1}$$

(R is the diameter; t - the thickness; M_x, M_y, κ_x, κ_y - moments and curvatures in the perpendicular planes passing through the axis of the cylinder, **i**, **j** are unit vectors) we can represent the results of the investigations in [4] in the following way.

If we consider the vector **M** as being analogous to the stress vector **P**, and the vector **N** analogous to the strain vector Э, then the initial yield curve in this case will be a circle with radius π. In the elastic material

$$\mathbf{P} = \pi Э \tag{4.2}$$

In the process of simple loading the yield curve is changing so that at the end of the vector **P** it forms a symmetrical angle. The magnitude of this angle is given by

$$\frac{p_s}{p} = \frac{\cos\varphi}{1 - (2\varphi - \sin 2\varphi)/\pi} \tag{4.3}$$

For an additional loading at the end of simple loading for $\alpha \geqslant \omega$, where ω is given by

$$\operatorname{tg}\omega = \frac{\pi - 2\varphi - \sin 2\varphi}{\pi - 2\varphi + \sin 2\varphi}\operatorname{tg}\varphi \tag{4.4}$$

the relationship between **P** and is given by the usual Hencky-Nadai theory, where the elastic and secant moduli satisfy

$$E = \pi, \qquad E_s = \pi - 2\varphi + \sin 2\varphi \tag{4.5}$$

In this work, the relationship between δ**P** and for additional loading in directions

$$-\varphi \leqslant \alpha \leqslant \omega$$

is found.

It is interesting to note the following circumstance. If **N** is analogous to **P**, and **N** π - **M** to $\mathcal{E}^p$, then it is easy to verify that during the additional loading at the end of a simple loading (1.2) is valid, where $A = p_s = 1$ and ϕ is constructed by the exterior tangents method. It is also easy to show that for the additional load from any state of stress formulas (1.2) are valid, where ϕ and ψ have to be considered as determined. The investigations of the dependence of ϕ and ψ on the loading path is somewhat complicated, as it was for the case considered in the previous section. We think, however, that the exterior tangents method should be also applicable in this case.

On the basis of the above, the following conclusions can be made.

(a) For an additional loading at the end of simple loading the Batdorf-Budiansky and Sanders theories coincide. This similarity is expected to exist for a broader class of loading paths.

(b) For plane loading paths and with condition (1.1) the resulting relations obtained from these two theories appear to be a special case of (1.2).

(c) If in Rabotnov's model we will consider the vector **N**π - **M** to be analogous to the plastic strain vector and the vector **N** to the stress vector, then the resulting relations appear to be also a special case of (1.2).

(d) It was shown in [3] that the second expression in (1.2), for $\phi = \psi$ and with the assumption of the applicability of the exterior tangents method, is identical with the Hencky-Nadai deformation theory. Thus, for an additional loading at the end of a simple loading, the Batdorf-Budiansky theory, Sanders theory and the resulting relations in Rabotnov's model coincide with the Hencky-Nadai theory for $\alpha \geqslant \phi$. If one succeeds to show the validity of the exterior tangents method for arbitrary loading paths in the Batdorf-Budiansky theory and for Rabotnov's model, (which seems to be very probable), then for all these theories one can prove the same as it was proved for the Sanders theory, viz. that they coincide with the Hencky-Nadai theory for the additional loads $a \geqslant \phi$ from an arbitrary state of stress in which $\phi = \psi$.

Quite instructional appears to be the fact that the old deformation theory, whose shortcomings under the conditions of the smoothness of the yield surface made it physically unreliable and thus had a considerable influence on the development of new approaches, precisely from the point of view of new concepts finds its place in the system of general relationships of plasticity.

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